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ON SLOW VISCOUS FLOW

by

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1. Introduction. The flow of a viscous fluid past an obstacle at low Reynolds Number has been the subject of numerous investigations. A rigorous analysis of the motion of the fluid requires the solution of a non-linear problem but various attempts have been made to formulate an equivalent[1] linear problem. The work presented here is a discussion of a new[2] linearization of this problem which is based on a conjecture rather than on any formal procedure such as a perturbation process. The analyses of several specific boundary value problems using this linearization are presented and the results are compared to those of the classical theories of Stokes [1] and Oseen [2]. The results reported here are in better agreement with the physical facts than those given by the Stokes or Oseen theories. We shall compare the point of view adopted in these classical methods with that adopted in ours as the analysis proceeds.

The actual details of the various problems treated here were carried out by several people at Brown University; the explicit references are included in the appropriate section. However the author is especially indebted to Professor G. W. Morgan

1. "Equivalent" in the sense of yielding a good approximation to the rigorous solution.

2. This linearization was actually used earlier on a specific problem by J. A. Lewis and the author [3].

who provided both the material and presentation of section 6.

2. The slow viscous flow problem. The motion of a viscous incompressible fluid is governed by the familiar laws requiring the conservation of momentum and mass. For an isotropic homogeneous medium of constant viscosity (i.e., μ independent of thermodynamic state) these laws take the form

$$u_j u_{i,j} + \rho^{-1} p_{,i} = \nu u_{i,jj} \tag{2.1}$$

and

$$u_{j,j} = 0. \tag{2.2}$$

Here, u_i, p, ρ, ν, are respectively the velocity, pressure, density, and kinematic viscosity. The differentiations are performed with regard to the physical coordinates x_i'.

The boundary conditions which are typical of some of the problems of physical interest require that the velocities be specified on the boundary curves of the region under consideration. The type of problem associated with such conditions is that in which we are interested here.

The differential equations can be put in a dimensionless form by introducing the following substitutions: $w_i = u_i/u_o$, $x_i = x_i' u_o/\nu$, $\sigma = p/\rho u_o^2$. Here, u_o is a characteristic velocity of the problem, e.g., the free stream velocity. Equations (2.1) and (2.2) then take the form

$$w_j w_{i,j} + \sigma_{,i} = w_{i,jj} \tag{2.3}$$

$$w_{j,j} = 0 \tag{2.4}$$

where the differentiations are now performed with regard to the x_i. This is the form of the equations which we shall find convenient but, in order that we may compare the classical analyses of this problem, we note that another formulation is readily obtained when one also introduces a length a which is characteristic of the geometry of the problem. We can then define $\varepsilon = u_0 a/\nu$, $\xi_i = x_i'/a$, $v_i = u_i/u_0$, and $\tau = p/\rho u_0^2$. Then Eqs. (2.1) and (2.2) take the form

$$\varepsilon v_j v_{i,j} + \tau_{,i} = v_{i,jj} \tag{2.5}$$

$$v_{j,j} = 0 \tag{2.6}$$

where the differentiations are taken with respect to the ξ_i.

Stokes, who was concerned with the flow of an otherwise undisturbed uniform stream past a solid motionless sphere, introduced a perturbation procedure using ε (the Reynolds number) as the expansion parameter. That is, he used the representations

$$v_j = \sum_{n=0}^{\infty} V_{jn}(\xi_p)\varepsilon^n, \quad \tau = \sum_{n=0}^{\infty} \tau_n(\xi_p)\varepsilon^n$$

and obtained a sequence of linear problems for the determination of the V_{jn} and τ_n when these series were introduced into Eqs. (2.5) and (2.6) and into the boundary conditions. When ε is sufficiently small, the results of such an analysis are in excellent agreement with the experimental observations (Fig. 1) for the flow past a sphere. However, when this procedure is attempted for certain two dimensional problems, it is found that no solutions of the linear problems so formed can exist. Specifically, those problems in which we require that $u_i = 0$ on a closed

curve and $u_1 \to u_0$, $u_2 \to 0$ as the distance from this curve increases do not admit a perturbation series of this type. This fact led Oseen to modify the Stokes' procedure somewhat. Essentially, using the notation of Eqs. (2.3) and (2.4), he developed the velocity components as

$$w_1 = 1 + w_1' + \ldots$$

$$w_2 = w_2' + \ldots ,$$

where the primed quantities are of "higher order"[3]. That is, he wrote the velocity field as the undisturbed velocity plus the perturbation induced by the obstacle. In each of these procedures, one computes only the terms of lowest order and these give results whose macroscopic features are reasonably accurate approximations to the rigorous result for $\varepsilon \leq 1$.

An alternative way of viewing these foregoing procedures (where only the first term of the expansion is found) does not require the formal introduction of a perturbation process or successive approximation scheme. One may consider an approximation technique where, in Eq. (2.1), we replace u_j by zero (for the Stokes model) and $u_0 \delta_{1j}$ (for the Oseen model). In the notation of Eq. (2.3) then, we have

$$c_j w_{i,j} + \sigma_{,i} = w_{i,jj} \tag{2.7}$$

3. The detailed argument can be found in [2]. It is clear, however, that w_1' must take on the value -1 on the obstacle and hence that w_1' is not strictly of higher order. Thus, Oseen's formulation can be taken as the basis of a successive approximation scheme but not as a formal perturbation procedure in, say, ε.

where c_j = (0,0) or (1,0) for the two models in question.

In order to motivate the forthcoming suggestion for a modification of these linearizations it is convenient (but not necessary) to consider the two dimensional problem. When we restrict ourselves in this way, we can write $w_1 = \psi_y$, $w_2 = -\psi_x$ and Eq. (2.3) implies that

$$\Delta\Delta\psi = \psi_y \; \Delta\psi_x - \psi_x \; \Delta\psi_y. \tag{2.8}$$

When we use Eq. (2.7) instead of (2.3) we obtain

$$\Delta\Delta\psi = c_1 \; \Delta\psi_x. \tag{2.9}$$

Let us now consider obstacles with a sharp leading edge which is positioned at the origin. The stream function ψ which is to satisfy the boundary conditions and Eqs. (2.8) or (2.9) (depending on whether we choose the exact problem or the approximate one) must have a branch point at the origin. One sees this in the following way. Let the boundary curve y(x) be an analytic function near and at the sharp leading edge and consider the function u_1 [x,y(x)] on this curve and on its analytic continuation into the flow field. Then on the boundary, u_1 [x,y(x)] is identically zero but on the continuation it is not. It follows that u_1 is not analytic at the origin. That is ψ must behave like r^n near[4] r = 0 and n must lie between the values 3/2 and 2, $[r = (x_1^2 + x_2^2)^{\frac{1}{2}}]$. For a cuspital edge (see [3]) n is 3/2. This implies that the contributions of the fourth derivatives on the left of Eq. (2.8) [or (2.9)] are more strongly singular than those terms on the right. This further implies that ψ is a function whose behavior near r = 0 is predominantly that of a

4. The singularity could be worse than this, depending on the leading edge shape.

biharmonic function. That is to say, if we subtract from ψ this singular biharmonic function the biharmonic term will be more highly singular than ψ - D and, in fact, $\psi/D \to 1$ as $r \to 0$. Here D is the biharmonic function of order r^n just mentioned.

Similarly, we can argue that as $r \to \infty$, the viscous terms are not of importance and ψ should be like a non-viscous flow. That is, ψ should become harmonic as $r \to \infty$. Note that either Eq. (2.8) or (2.9) admits such harmonic solutions (anywhere). The important fact to note now is that while each equation together with the boundary conditions is capable of "generating" a function ψ which is predominantly biharmonic near $r = 0$ and harmonic as $r \to \infty$, their behaviors in the intermediate region are not identical and there is little reason to hope that each equation will imply a ψ which continues from (say) a given biharmonic behavior near $r = 0$ to the same harmonic function at large r. However, it is to be hoped that if c_1 were neither 1 nor 0, but had a value corresponding to some appropriately weighted average of the velocity, the differences in the values of the solutions of the two equations (2.8) and (2.9) would vanish on the average. If this should occur, then Eq. (2.9) and its boundary conditions could generate a solution which was essentially correct at large r, and near the front of the obstacle, but which was not necessarily a very accurate approximation in between.

Since the boundary conditions are to be applied where the solutions are hypothetically correct, however, the macroscopic features of the prediction would be acceptable.

We should note that if this sort of an "average

equation", (2.9), should be acceptable as a substitute for (2.8), it is probable that c_1 would be a function of the important dimensionless parameter appearing in such problems, i.e., the Reynolds number ε.

Our conjecture then is that there may exist a function $c(\varepsilon)$, $0 < c < 1$, such that the solution of Eq. (2.8) under suitable boundary conditions can be "replaced" by a solution of Eq. (2.9) with c_1 replaced by $c(\varepsilon)$. If the conjecture is to be useful it is necessary, of course, that c depend <u>only</u> on ε.

In the succeeding sections we shall present the manner in which we chose the presumably correct value of c for moderately small ε and the comparison of the results of this theory with either experimental observation or the solution of the corresponding non-linear problem. It is to be noted that although we have based our conjecture on a motivation associated with a sharp edged obstacle, we shall "test" it on a broader group of problems.

3. <u>The flow past a long flat plate</u>. The classical problem associated with the laminar flow of a viscous fluid past a semi-infinite flat plate is the problem which gave rise to the investigations presented in this paper. In 1948, C. C. Lin and the author attempted a calculation of the flow in the neighborhood of the leading edge of such a plate [4]. The critical argument of that investigation required that the region in which the leading edge solution was valid overlap the region in which the Blasius solution was valid. However, there seems to be no convincing argument that such an overlap region exists. Before the doubtful validity of this result was noticed, however, the same

problem was considered [3] using the Oseen formulation and at that time the conjecture of this paper was introduced. In order to force the result of this linearized problem to match the leading-edge result, the number c was given the value .35. The solution of the linearized problem is outlined in the appendix (the details can be found in [3]) and the result is given by

$$\begin{aligned}\Psi(x,y) &= 2\xi[\eta \operatorname{erf}(\eta c^{1/2}) - (\pi c)^{-1/2}(1 - e^{-cy^2})] \\ &= \xi\, g(\eta) \qquad (3.1)\end{aligned}$$

where $\xi + i\eta = (x + iy)^{1/2}$ and Ψ is the stream functions defining the flow.

When $c = .35$, the velocity gradient at the plate is the same as that found by Blasius but the behavior for large η (and any ξ) differs from that of the Blasius solution. In fact, if we denote the Blasius result by Ψ_B and ours by Ψ then

$$\Psi_B = \xi f(\eta) \qquad (3.2)$$

and asymptotically (for large η)

$$\begin{aligned}\Psi_B &\sim \xi\,(2\eta - 1.72 + \dots) \\ \Psi &\sim \xi\,[2\eta - 2(\pi c)^{-1/2} + \dots] \qquad (3.3)\end{aligned}$$

These asymptotic developments agree only when $c = .43$.

It would now appear that one might choose $c = .35$ if the Lin-Carrier result were believed to be valid, or $c = .43$ if it seemed appropriate to match the large η behavior of the linearized answer and the Blasius result. The latter can be argued

against by noting that the Blasius result is the first term of an asymptotic development in ξ and hence is valid only for $\xi >> 1$ whereas one can anticipate the validity of our linearized result only for small ξ. The argument against the former has already been noted.

Before the foregoing discrepancy was noted, we proceeded to investigate the problems presented in sections (4), (5), and (6). For moderataly low values of the Reynolds number, the results for the problems of these sections provide excellent agreement between the non-linear theory or experimental evidence and the c = .43 result. This fact, however, does not justify using .43 (or any other c) at high Reynolds number.

In order to determine whether any value of c might be appropriate for the high Reynolds number problem of this section, the flow field for the semi-infinite plate problem is being computed from the non-linear formulation, via the relaxation method. The investigation has not yet been completed but it is clear that the local linearized prediction for c = .43 will not agree with the non-linear prediction. This leads to the amusing situation where the choice c = .43 which is very good at small and moderate values of the Reynolds number resulted from the analysis of an infinite Reynolds number problem to which it is not particularly appropriate.

4. <u>The flow past the cylinder and the sphere</u>. The application of the present theory to the determination of the flow of an otherwise undisturbed uniform stream past an obstacle is a rather simple matter when the corresponding Oseen result is

available. A discussion of the flow past a sphere will illustrate this fact clearly and the extension to other obstacles (including, of course, the cylinder) is a straightforward matter. In this discussion it is most convenient to use the notation of Eq. (2.5). In our linearized theory the momentum equation takes the form

$$c\varepsilon v_{i,1} + \sigma_{,i} = v_{i,jj}. \tag{4.1}$$

The Oseen theory differs from this in that c is replaced by 1. The boundary value problem requires the solution of Eqs. (2.6) and (4.1) subject to the conditions $v_j = 0$ on $r^2 = x_j x_j = 1$ and $v_j \longrightarrow \delta_{1j}$ as $r \longrightarrow \infty$.

We now denote the tangential velocity component associated with the solution of the Oseen problem ($c = 1$) by $v(\varepsilon,r,\theta)$ and that associated with the modified linearization as $V(\varepsilon,r,\theta)$. It is readily seen that $V(\varepsilon,r,\theta) \equiv v(c\varepsilon,r,\theta)$. The other velocity components and the pressure may be related in the same way. The friction drag on the sphere (according to the present theory) is given by

$$D = 2\pi\mu\, u_o a \int_0^\pi V_r(\varepsilon,1,\theta)\, \sin^2\theta\, d\theta$$

$$= 2\pi\mu\, u_o a \int_0^\pi v_r(c\varepsilon,1,\theta)\, \sin^2\theta\, d\theta$$

and the corresponding drag coefficient is given by

$$C_D = (\mu/\rho u_o a) \int_0^\pi v_r(c\varepsilon,1,\theta)\, \sin^2\theta\, d\theta.$$

However, the friction drag coefficient C_D', given by the Oseen

theory for Reynolds number $c\varepsilon$ is

$$C_D' = \frac{1}{c\varepsilon}\int_0^{\pi} v_r(c\varepsilon,1,\theta)\,\sin^2\theta\,d\theta$$

so that

$$C_D = c\,C_D'(c\varepsilon). \tag{4.2}$$

The contributions to the drag of the pressure variation over the sphere are related in the same way. Thus, Eq. (4.2) expresses the drag coefficient relationship for the two theories for the gross drag or for either contribution. The same formula is valid for the cylinder or any other obstacle family characterized by a single length parameter.

In view of this result we may use the results of drag coefficient calculations of previous authors to find those associated with our theory. Figure (4.1) indicates the results of this calculation for the cylinder using the figures obtained by Bairstow [8] and also indicate some experimental evidence concerning the drag. Figure (4.2) indicates the corresponding evidence for the sphere. It is clear that the present theory with $c = .43$ gives a much more accurate prediction of the drag than do previous accounts of the matter.

5. The finite flat plate. In this section the low Reynolds number flow past a flat plate will be discussed. The formulation of the problem differs from that of the appendix only in the respect that the boundary condition $\psi_y = -1$, on $y = 0$, $x > 0$, applies only when $x < a$. Here a is the length of the plate in units ν/u_0.

The analysis of this problem follows that of the appendix up to Eq. (3.4). At that stage of the preceedings, it is convenient to note that

$$\bar{\sigma}_y(\xi,a) = \frac{\bar{f}(\xi)}{2(\xi-ik)^{1/2}[(\xi+ik)^{1/2}+(\xi+i)^{1/2}]} = \bar{f}(\xi)\bar{h}(\xi) \quad (5.1)$$

and to apply the inversion integral to each side of this equation using the convolution theorem to evaluate the right hand side. We obtain[5]

$$\psi_y(x,0) = \int_0^a f(\tau)h(x-\tau)d\tau$$

where

$$h(x) = \left(\frac{1}{\pi x}\right) \int_0^{cx/2} e^u\, k_o|u|du.$$

Since $f(x)$ must have a singularity of order $\frac{1}{2}$ at $x = 0$ and at $x = a$, it is convenient to write

$$f(x) = g(x)/\sqrt{x(a-x)}$$

$$\psi_y(x,0) = \int_0^a g(\tau)[h(x-\tau)/\sqrt{\tau(a-\tau)}]d\tau.$$

The substitutions $\tau = a\sin^2\theta$, $x = a\sin^2\theta$, lead to the equation

$$\psi_y(x,0) = 1 = \int_0^{\pi/2} g(a\sin^2\theta)\; h\,(a\sin^2\theta - a\sin^2\theta)d\theta.$$

This integral equation for g may now be solved either by finding the coefficient of the expansion

$$g(a\sin^2\theta) = \Sigma\, \alpha_n\, \theta^n$$

5. A more detailed account of this analysis can be found in [6].

or by a numerical process. It turns out that an excellent approximation to g for moderate values of a is given by

$$g(a \sin^2\theta) = A(a) + B(a)(\theta - \pi/4).$$

The quantities A and B are plotted in Fig. (5.1). In particular, the drag is given by $\mu U_o \pi A/2$. The flow past a finite flat plate according to the non-linear theory, Eqs. (2.1) and (2.2), has been found by Munier [7]. Using the relaxation technique the flow field [i.e., $u_1(x,y)$, $p(x,y)$] has been computed for a = 4 and, in particular, the velocity gradient at the plate has been recorded. This velocity gradient as deduced from each of these theories is given in Fig. (5.2). The agreement of the results of the linearized theory with those of the non-linear theory is surprisingly good.

6. <u>The flow in a wedge shaped region</u>. In this section we shall present the linearized analysis of the converging flow in a wedge shaped region and compare the results to the exact solution obtained by Hamel. This problem is not in the category for which the linearization is designed. However, it is of interest to see how well the flow is predicted by this theory.

We start from Eqs. (2.1) and (2.2) and use the radial coordinates r,θ. The flow proceeds towards the origin in the wedge shaped region $-\theta_o < \theta < \theta_o$. We look for solutions $\psi = \psi(\theta)$ corresponding to which the radial velocity is given by $u(r,\theta) = r^{-1}\psi'(\theta)$ and the circumferential velocity vanishes everywhere. We denote by ε (the Reynolds number) the quantity

$UR\rho/\mu$ where $u(r,\theta) = -U\frac{R}{r}$ and $u(r,\theta)$ can be written

$$u(r,\theta) = -(UR/r)\chi'(\theta).$$

Equation (2.1) and (2.2) now yield

$$\Delta\Delta\chi + \epsilon r^{-1}\chi'(\Delta\chi)_r = 0. \tag{6.1}$$

Since χ is a function of θ only

$$\chi^{IV} + 4\chi'' - 2\epsilon\chi'\chi'' = 0. \tag{6.2}$$

Note, however, that in Eq. (6.1), the quantity χ'/r corresponds to the coefficient v_j in Eq. (2.5) and hence corresponds roughly to the term which is replaced by unity in the Oseen treatment and by c in ours. However, it is profitable to retain the information that u behaves like 1/r and to replace χ' by c retaining the 1/r contribution. This, clearly, is in the spirit of our approximation technique since $\chi'(\theta)$ varies from unity to zero and has some "average", c. Our linearized equation then takes the form

$$\chi^{IV} + 4\chi'' - 2c\epsilon\chi'' = 0. \tag{6.3}$$

6.1. <u>The exact result</u>. Equation (6.2) can be integrated twice to yield

$$\frac{(w')^2}{2} + 2w^2 - \frac{\epsilon}{3}w^3 + Cw = D \tag{6.4}$$

where C and D are arbitrary constants which must be determined and we have put $w = \chi'$. We have $\chi'(0) = 1$ and therefore $w(0) = 1$. Also, because of the symmetry about $\theta = 0$, we must have $w'(0) = 0$. These two conditions together with Eq. (6.4) give the following relation

$$2 - \frac{\varepsilon}{3} + C = D. \tag{6.5}$$

Hence (6.4) becomes

$$\frac{(w')^2}{2} + 2w^2 - \frac{\varepsilon}{3} w^3 + Cw = C - \frac{\varepsilon}{3} + 2. \tag{6.6}$$

To integrate Eq. (6.6) we rewrite it as follows:

$$\frac{(w')^2}{2} = \frac{\varepsilon}{3} (w - 1)[w^2 + (1 - \frac{6}{\varepsilon})w + (1 - \frac{3C}{\varepsilon} + \frac{6}{\varepsilon})]. \tag{6.7}$$

We now factor the expression in square brackets so as to obtain (6.7) in the form

$$(w') = \frac{2}{3} \varepsilon (w - 1)(w - a_1)(w - a_2) \tag{6.8}$$

where

$$a_1 = \frac{1}{2}\left\{(\frac{6}{\varepsilon} - 1) + \sqrt{(1 - \frac{6}{\varepsilon})^2 + \frac{4}{\varepsilon} (3C + 6 - \varepsilon)}\right\}$$

$$a_2 = \frac{1}{2}\left\{(\frac{6}{\varepsilon} - 1) - \sqrt{(1 - \frac{6}{\varepsilon})^2 + \frac{4}{\varepsilon} (3C + 6 - \varepsilon)}\right\}$$

Using Eq. (6.6) and the boundary condition of viscous flow that $w(\pm \theta_0) = 0$, we see that $C - \frac{\varepsilon}{3} + 2 > 0$ and hence that

$$(1 - \frac{6}{\varepsilon})^2 + \frac{4}{\varepsilon} (3\varepsilon + 6 - \varepsilon) > 0.$$

Thus a_1 and a_2 are real.

Equation (6.8) can now be integrated in terms of elliptic functions. We have from (6.8)

$$w' = \pm \sqrt{\frac{2}{3}}\, \varepsilon \sqrt{(w - 1)(w - a_1)(w - a_2)}\ . \tag{6.9}$$

Since $w(0) = 1$ and $w(\pm \theta_0) = 0$, we expect that w will range between 0 and 1 and that $w' \geq 0$ between $-\theta_0$ and 0 and $w' \leq 0$ between 0 and θ_0. Hence the positive sign in (6.9) holds for $\theta \leq 0$.

Let us integrate from $-\Theta_0$ to some non-positive angle Θ.

$$\sqrt{\frac{3}{2\varepsilon}}\int_0^w \frac{d\xi}{\sqrt{(\xi-1)(\xi-a_1)(\xi-a_2)}} = \int_{-\Theta_0}^{\Theta} d\eta = \Theta + \Theta_0 \qquad (6.10)$$

$$\Theta \leq 0$$

Now a_1 and a_2 still contain an unknown constant C which must be evaluated from the condition that $w = 1$ and $\Theta = 0$. Hence

$$\sqrt{\frac{3}{2\varepsilon}}\int_0^1 \frac{d\xi}{\sqrt{(\xi-1)(\xi-a_1)(\xi-a_2)}} = \Theta_0. \qquad (6.11)$$

Upon integration, Eq. (6.11) will furnish, in terms of elliptic functions, a formula relating C, ε, and Θ_0. Substituting this into (6.10) and integrating we obtain the solution of the problem.

Tables of elliptic integrals give the following relation

$$\int_w^{\beta} \frac{d\xi}{\sqrt{(\alpha-\xi)(\beta-\xi)(\xi-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\, sn^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\beta-\gamma}\,\frac{\beta-w}{\alpha-w}}\;\sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}\right)$$

(6.12)

$\alpha > \beta > w > \gamma$. The inequality is satisfied, as we shall soon see, if we let $a_1 = \alpha$, $1 = \beta$, $a_2 = \gamma$. Now

$$\int_0^w = \int_0^1 - \int_w^1 = \sqrt{\frac{2\varepsilon}{3}}\,\Theta_0 - \int_w^1$$

by using Eq. (6.11). Hence, using (6.10), we have:

$$-\int_w^1 \frac{d\xi}{\sqrt{(a_1-\xi)(1-\xi)(\xi-a_2)}} = \sqrt{\frac{2\varepsilon}{3}}\,\Theta. \qquad (6.13)$$

We now consider three special cases corresponding to different ranges of the Reynolds number, ε.

(a) $\varepsilon < 6$; special case $\varepsilon = 1$.

For $\varepsilon = 1$, the roots a_1, a_2 are

$$a_1 = \frac{1}{2}\left\{5 + \sqrt{45 + 12c}\right\}$$

$$a_2 = \frac{1}{2}\left\{5 - \sqrt{45 + 12c}\right\}$$

We previously saw that $C > \frac{\varepsilon}{3} - 2$. Hence $a_1 > 5$, $a_2 < 0$, and the inequality in (6.12) is satisfied.

(b) $\varepsilon = 6$.

In this case the roots are

$$a_1 = \sqrt{\frac{C}{2}}, \qquad a_2 = -\sqrt{\frac{C}{2}}.$$

To estimate the values of a_1 and a_2 we make use of another inequality containing C. Since $u(r,\Theta)$ will have a minumum at $\Theta = 0$, $w(\Theta)$ will have a maximum there, and hence $w''(0) < 0$. Now the first integral of Eq. (6.3) is

$$w'' + 4w - \varepsilon w^2 + C = 0.$$

This equation, together with $w(0) = 1$ and $w''(0) = 0$ gives $C > \varepsilon - 4$.

Hence, we have for $\varepsilon = 6$:

$$a_1 > 1, \qquad a_2 < -1$$

and the inequality in (6.12) is again satisfied.

(c) $\varepsilon > 6$, special case $\varepsilon = 120$.

The procedure of cases (a) and (b) leads to $a_1 > 1$ and $a_2 < -2$, so that our integration formula again holds.

Since Eq. (6.11) gives θ_0 in terms of ε and C, the procedure followed in the numerical computation is to pick a value of C for a given ε and to find the corresponding θ_0. It is due to this that the calculations for the three values of ε were not carried out for one and the same wedge angle, but rather for three angles which are only approximately equal.

Wall Drag. The formula for the drag can immediately be obtained from (6.6) by putting w = 0. This gives

$$w'(-\theta_0) = \sqrt{\frac{2}{3}}\sqrt{3C - \varepsilon + 6}\,. \tag{6.14}$$

The drag is readily obtained from this by recalling that $u(r,\theta) = -\frac{U_\omega}{r} w(\theta)$.

6.2. The linearized result. In this section we denote the radial velocity component by $1 + f(\theta)$ and Eq. (6.3) becomes

$$f''' + (4 - 2c\varepsilon)f' = 0 \tag{6.15}$$

and the boundary conditions are $f(0) = 0$ and $f(\pm\theta_0) = -1$. We denote $4 - 2c\varepsilon$ by s^2 and distinguish the three cases, $s^2 < 0$, $s^2 = 0$, $s^2 > 0$. The appropriate solutions of Eq. (6.15) are

(a) $s^2 > 0$,

$$w = 1 + f = \frac{\cos s\theta - \cos s\theta_0}{1 - \cos s\theta_0} \tag{6.16}$$

(b) $s^2 = 0$,

$$w = 1 + f = 1 - \frac{\theta^2}{\theta_o^2} \tag{6.17}$$

(c) $s^2 < 0$,

$$w = 1 + f = \frac{\cosh s'\theta - \cosh s'\theta_o}{1 - \cosh s'\theta_o} \tag{6.18}$$

where $s' = - s$.

The velocity gradients at the wall can be found immediately. They are:

(a) $\varepsilon < \frac{2}{c}$; $\qquad w'(-\theta_o) = \dfrac{s \sin s\theta_o}{1 - \cos s\theta_o}$ (6.19)

(b) $\varepsilon = \frac{2}{c}$; $\qquad w'(-\theta_o) = \dfrac{2}{\theta_o}$ (6.20)

(c) $\varepsilon > \frac{2}{c}$; $\qquad w'(-\theta_o) = \dfrac{s' \sinh s'\theta_o}{1 - \cosh s'\theta_o}$. (6.21)

6.3. <u>Boundary layer solution</u>. As a matter of interest we consider the boundary layer solution of this problem. We start from Eq. (6.3) and integrate it once to obtain

$$w'' + 4w - \varepsilon w^2 + C = 0. \tag{6.22}$$

In the exact solution we determined C from the condition that $w(0) = 1$. In the boundary layer solution we demand only that w approach unity asymptotically as we go away from the wall. Hence C can only be determined in the limit as $\varepsilon \longrightarrow \infty$.

In the interior, we expect w and its derivatives to be of order one; hence, for sufficiently large ε, Eq. (6.22) becomes approximately

$$- \varepsilon w^2 + C = 0. \tag{6.23}$$

Since w is to be one on $\Theta = 0$, we see that C must be equal to ε within the order of approximation of the boundary layer theory.

Following the usual procedure in boundary layer theory we transform to a new independent variable

$$\varphi = \varepsilon^{1/2}(\Theta_0 - \Theta), \qquad w(\Theta) = \tau(\varphi) \tag{6.24}$$

in which τ and its derivatives are all of order one. We have

$$\varepsilon\tau'' + 4\tau - \varepsilon\tau^2 + \varepsilon = 0. \tag{6.25}$$

For ε approaching infinity this gives the approximate boundary layer equation

$$\tau'' - \tau^2 + 1 = 0. \tag{6.26}$$

The boundary conditions $w(\Theta) = 1$ and $w(\Theta_0) = 0$ transform to

$$\lim_{\varphi \to \infty} \tau = 1 \quad \text{and} \quad \tau(0) = 0. \tag{6.27}$$

One integration of (6.26) gives

$$\frac{(\tau')^2}{2} - \frac{\tau^3}{3} + \tau = D. \tag{6.28}$$

To evaluate D we apply the condition that $\tau \to 1$ as $\varphi \to \infty$, and hence $\tau' \to 0$ as $\varphi \to \infty$. This gives $D = 2/3$. We can now write

$$(\tau')^2 = \frac{2}{3}(1 - \tau)^2(\tau + 2) \tag{6.29}$$

and

$$\int_0^{\tau} \frac{d\xi}{(1 - \xi)\sqrt{(2 + \xi)}} = \sqrt{\frac{2}{3}} \int_0^{\varphi} d\eta = \sqrt{\frac{2}{3}}\, \varphi. \tag{6.30}$$

Upon integration and substitution of w and Θ in terms of τ and φ this leads to the following solution:

$$\Theta_0 - \Theta = \sqrt{\frac{2}{\varepsilon}}\,[\tanh^{-1}\sqrt{\frac{w+2}{3}} - \tanh^{-1}\sqrt{\frac{2}{3}}]. \qquad (6.31)$$

The velocity gradient at the wall can be found from Eq. (6.29) by putting $\tau = 0$. Hence

$$(\tau'(0))^2 = 4/3$$

and

$$w'(-\Theta_0) = \frac{2}{\sqrt{3}}\sqrt{\varepsilon}\,. \qquad (6.32)$$

The most important test for applicability of the various approximate theories is probably to be found in the values they predict for the velocity gradient at the wall. An inspection of the table of results shows that, for the problem at hand, both the standard and the modified Oseen technique give results that are accurate to within a few percent for $\varepsilon = 1$ and $\varepsilon = 6$, but that the modified method is appreciably better. For ε of the order 100, the Oseen methods are quite inaccurate, but the boundary layer solution is already fairly good. It is interesting to note (see Fig. 6) that the velocity profile as calculated by boundary layer theory is very close to the exact profile in a region starting at the wall and extending to nearly one third the total channel width.

7. Conclusions. It is evident that, in the foregoing moderate Reynolds number problems, a successful prediction of the macroscopic feature of the flow are given by the linearized theory with $c = .43$. It can safely be anticipated that an

equally successful application can be anticipated for other flows of a similar nature. For more complicated problems, e.g., the flow past a plate of finite chord at non-zero incidence, no conclusion can be drawn until such flows have been investigated. Similarly, it is not clear whether the range of applicability can be increased by finding a $c(\varepsilon)$ for larger ε than those considered here.

8. Appendix. The boundary value problem of section (3) is:

$$\Delta\Delta\psi - c\Delta\psi_x = 0; \tag{2.9}$$

and

$$\psi(x,0) = 0; \quad \psi_y(x,0) = -1 \quad \text{for} \quad x > 0.$$

The solution as given in [3] is found by using Fourier transform. We define

$$\overline{\psi}(\xi,\eta) = \iint_{-\infty}^{\infty} e^{-i(\xi x+\eta y)}\psi(x,y)dxdy, \tag{8.1}$$

and note that ψ, ψ_y, and ψ_{yyy} are continuous across the plate. We denote by $f(x)$ the jumps in $\psi_{yy}(x,0)$. Using the foregoing, Eq. (2.9) becomes

$$[(\xi^2 + \eta^2)^2 + ic\xi(\xi^2 + \eta^2)]\overline{\psi} = -i\eta\overline{f}(\xi). \tag{8.2}$$

Equation (8.2) may conveniently be thought of as the limit as $k \to 0$ of the equation

$$(\xi^2 + \eta^2 + k^2)(\eta^2 + [\xi + i][\xi - ik])\overline{\psi} = -i\eta\overline{f}. \tag{8.3}$$

We may also note that the boundary condition on ψ_y at the plate is the limit as $\alpha \to 0$ of $\psi_y(x,0) = e^{-\alpha x}$ for $x > 0$. We may now

define
$$\bar{\varphi}(\xi,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\psi}(\xi,\eta)e^{i\eta y}\,dy$$

and obtain
$$\bar{\varphi}(\xi,y) = \frac{|y|\bar{f}[\exp\{-|y|(\xi^2+k^2)^{1/2}\} - \exp\{-|y|(\xi+i)^{1/2}(\xi-ik)^{1/2}\}]}{y\,2i(1-k)(\xi-ik)} \tag{8.4}$$

We may now define $\bar{\varphi}_y(\xi,0) = \bar{u}_o(x) \times \bar{u}_1(x)$ where $u_o(x) = e^{-\alpha x}$ where $x > 0$ and $u_o(x) = 0$ when $x < 0$.

We may now use the usual Wiener-Hopf arguments to find $\bar{f}(\xi)$ [the details are to be found in [3]] and find, when $k \to 0$, $\alpha \to 0$,
$$\bar{f} = 2i(i/\xi)^{1/2}. \tag{8.5}$$

We now can invert Eq. (8.4) using this form for $\bar{f}$ and we obtain
$$\psi(x,y) + y = 2\xi[\eta\,\mathrm{erf}\,(\eta\sqrt{c}) - (\pi c)^{-1/2}(1 - e^{-c\eta^2})\,]$$
$$= \xi g(\eta). \tag{8.6}$$

Here, $\xi + i\eta = (x + iy)^{1/2}$ and $\psi + y$ is the stream function for the complete flow. The bracket is denoted by $g(\eta)$ for convenience[6].

Consistent now with the motivation of section (1) we would like to choose c so that this solution agrees with the solution of the precise problem both near the plate leading edge and far from the plate. Our only information is that for large distances downstream of the leading edge ($x > 20$, say), the far

6. This explicit result (i.e., Eq. (8.6)) was not included in [3].

field solution is given by

$$\psi_B \sim \xi \; f(\eta) \tag{8.7}$$

where $f(\eta) \sim 2\eta - 1.72 + \ldots$. The remaining terms do not correspond to a harmonic function. The asymptotic behavior of $\psi = \xi g(\eta)$ becomes the same as that of Eq. (8.7) provided we choose $c = .43$. Thus, if any function $c(\varepsilon)$ is to accomplish the purpose outlined in section (1), its value as $\varepsilon \to \infty$ should be .43. In Fig. (8.1) the function $f'(\eta)$ and $g'(\eta)$ are plotted. One should note that $|f'/g' - 1|$ is never greater than .1, the most serious discrepancy occurring at the plate surface.

Table of results for section (6).

ε	θ_o radians	$w'(-\theta_o)$ exact solution	Standard Oseen	Modified Oseen $c = .43$	Boundary Layer
1	.295	6.63	6.67	6.62	--
6	.344	6.00	6.26	5.88	--
120	.279	13.11	15.79	10.47	12.65

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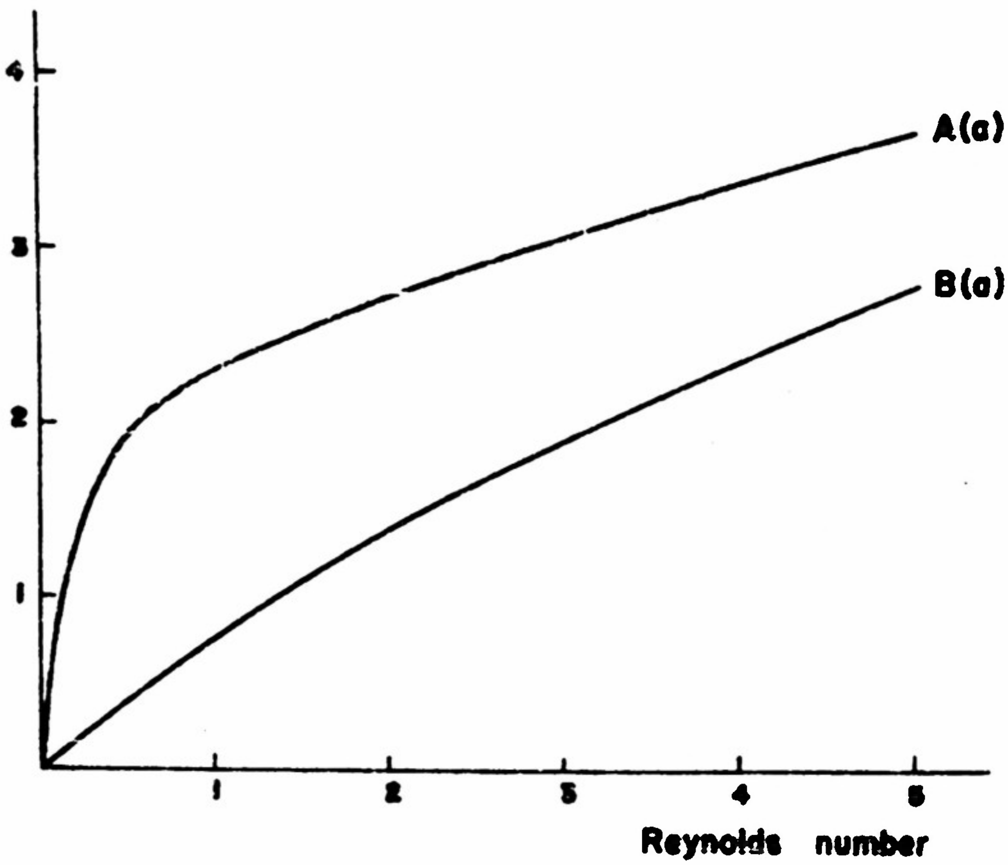

FIG. 5.1

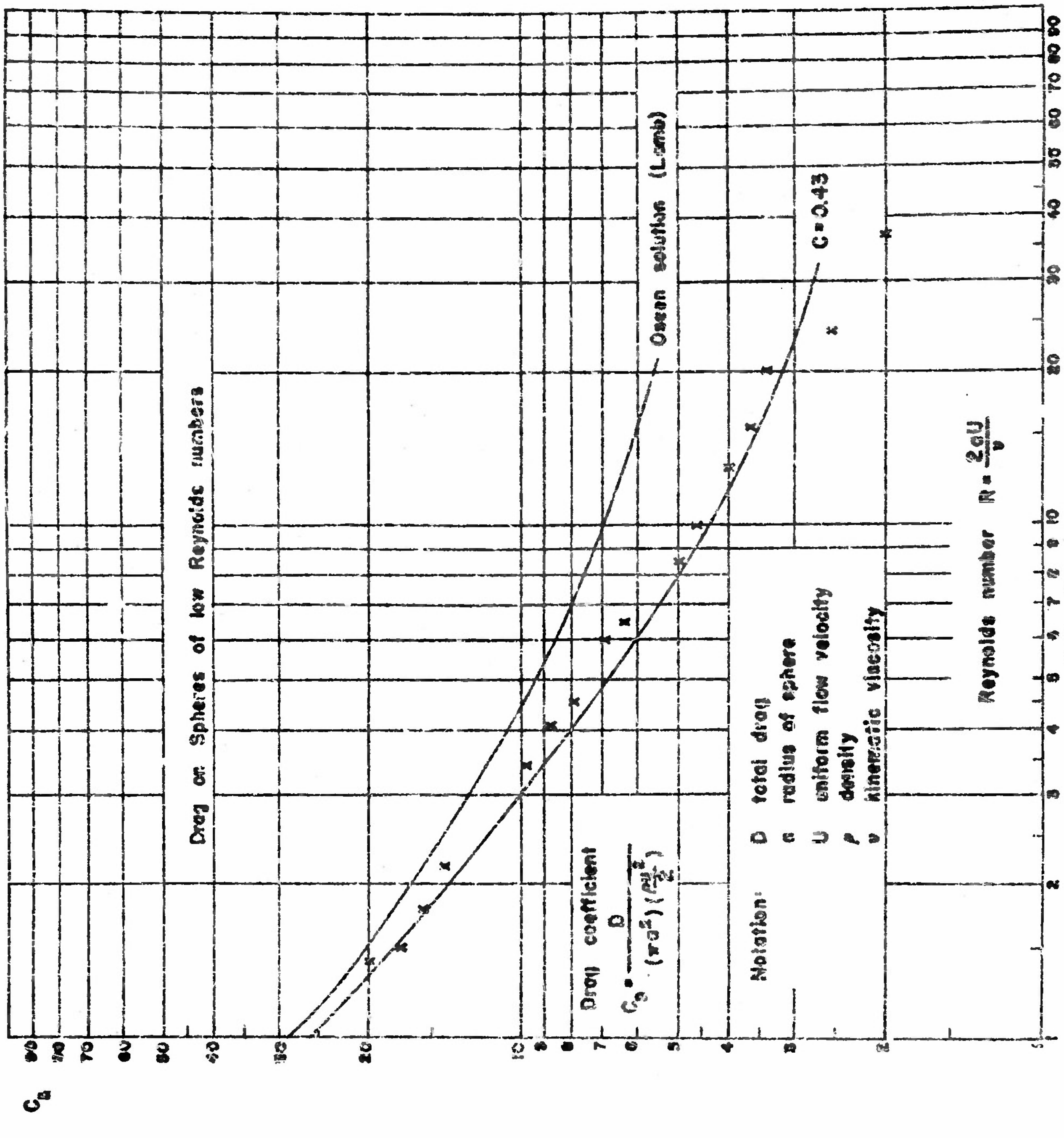
Drag on Spheres of low Reynolds numbers
Oseen solution (Lamb)
C = 0.43
Drag coefficient
$C_D = \frac{D}{(\pi a^2)(\rho U^2/2)}$
Notation:
D total drag
a radius of sphere
U uniform flow velocity
ρ density
ν kinematic viscosity
Reynolds number $R = \frac{2aU}{\nu}$
C_D
1
2
3
4
5
6
7
8
9
10
20
30
40
50
60
70
80
90

FIG. 4.1

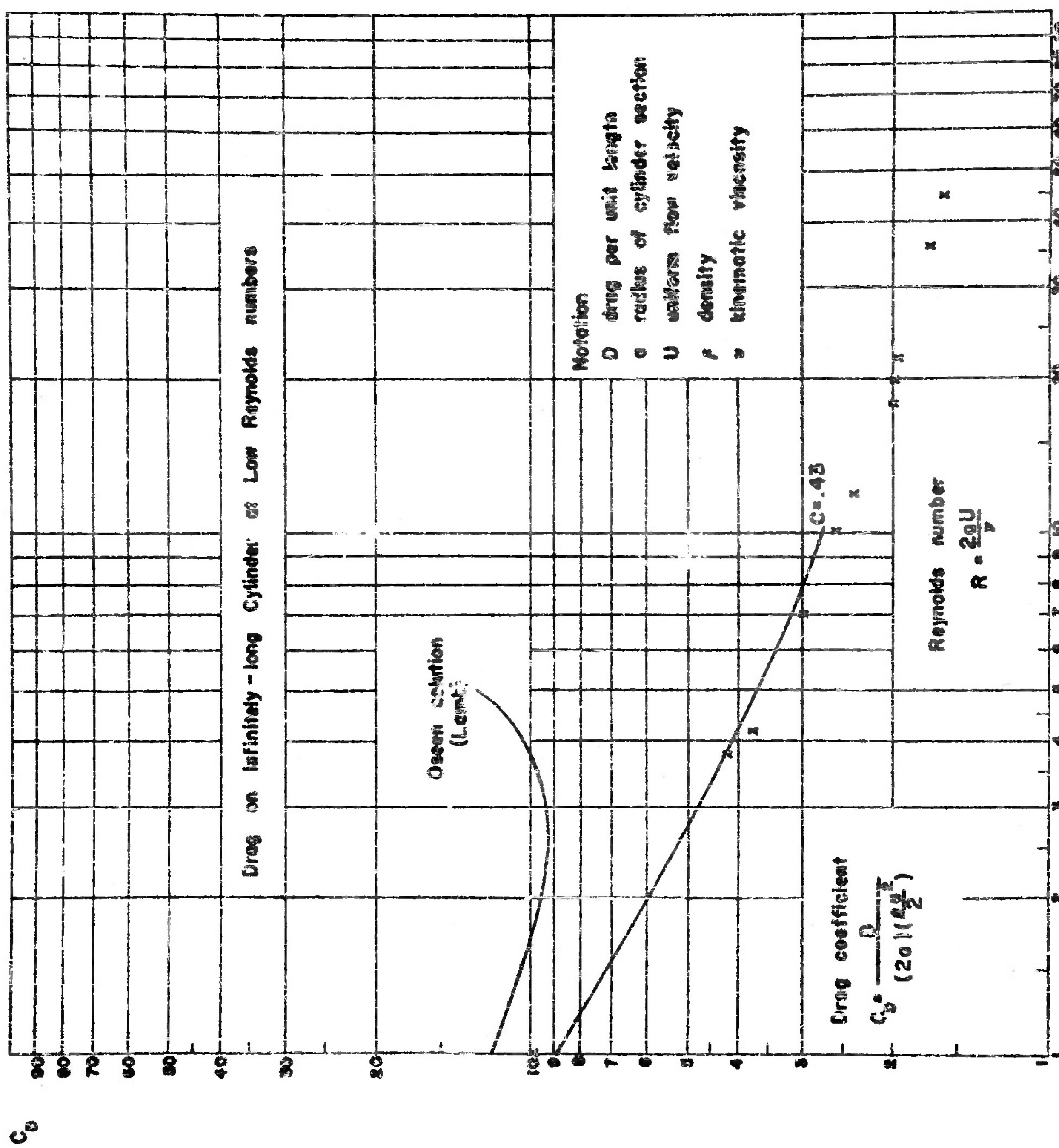
Drag on infinitely-long Cylinder at Low Reynolds numbers
Notation
D drag per unit length
a radius of cylinder section
U uniform flow velocity
ρ density
ν kinematic viscosity
Oseen solution
C=.43
Drag coefficient
$C_D = \frac{D}{(2a)(\frac{\rho U^2}{2})}$
Reynolds number
$R = \frac{2aU}{\nu}$
C_D

FIG. 4.2

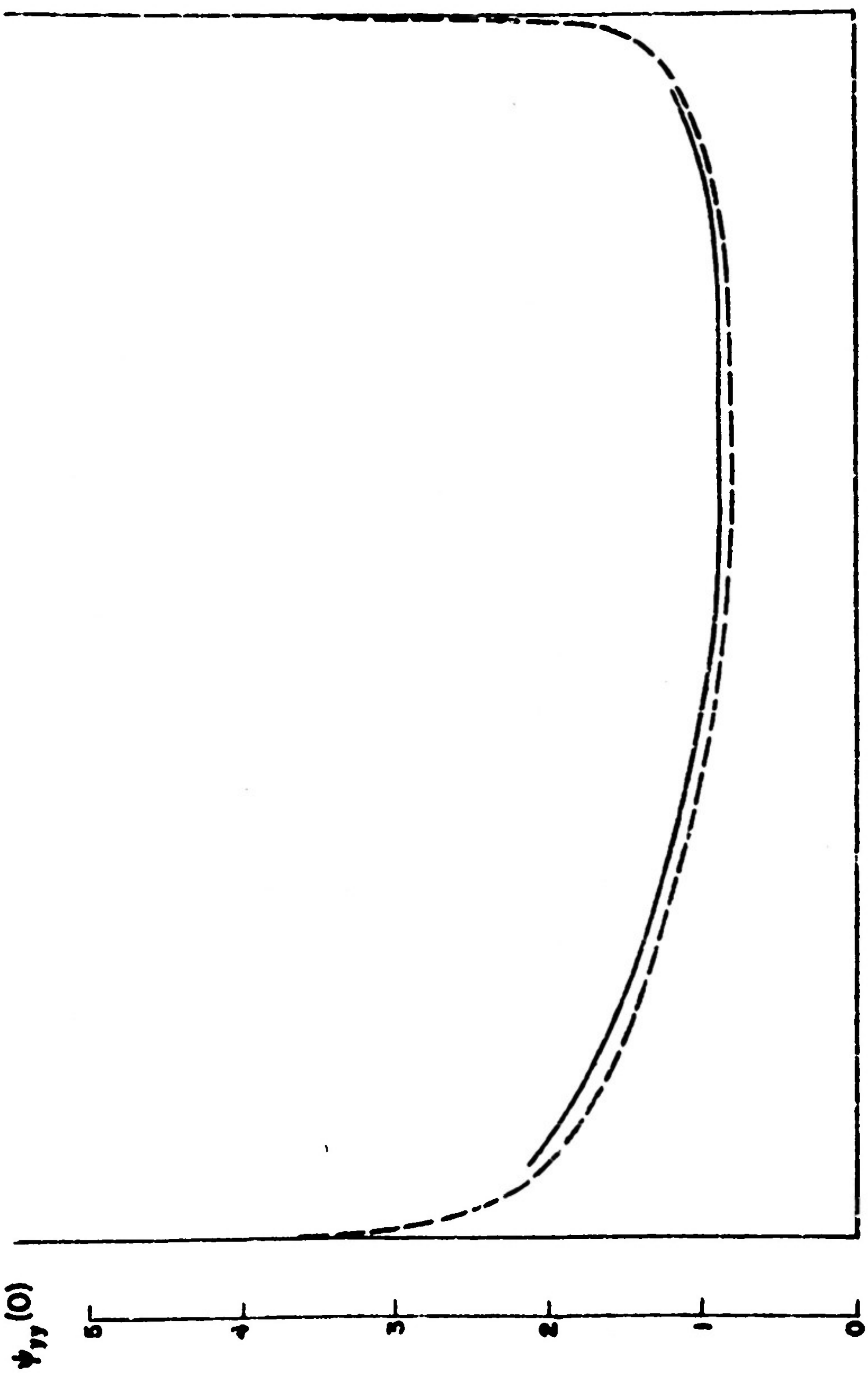

FIG. 5.2 Velocity gradient at the plate for $\alpha=4$. The dotted curve represents the linearized result. The solid curve is the finite difference non-linear result.

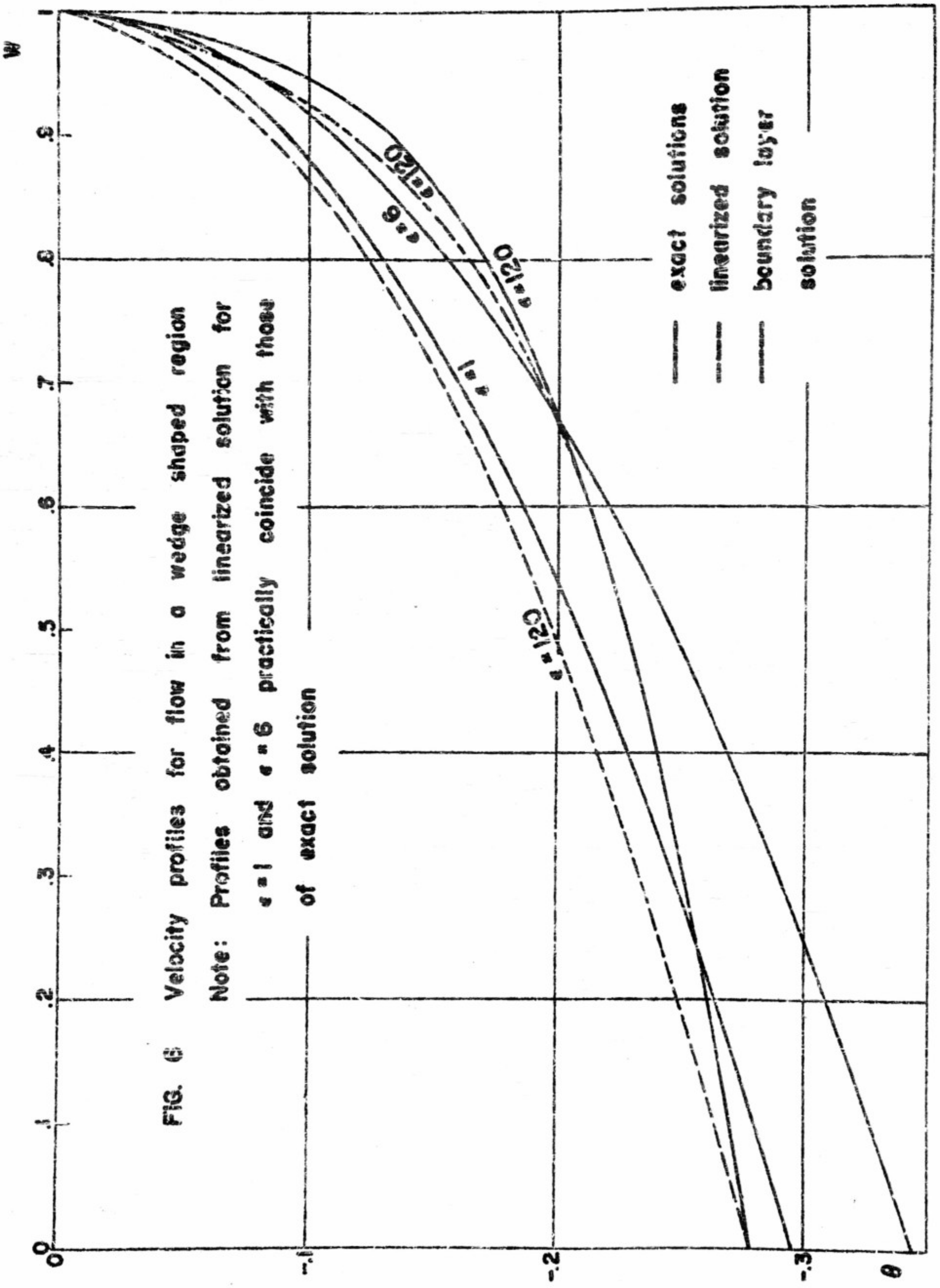

FIG. 6 Velocity profiles for flow in a wedge shaped region

Note: Profiles obtained from linearized solution for ε=1 and ε=6 practically coincide with those of exact solution

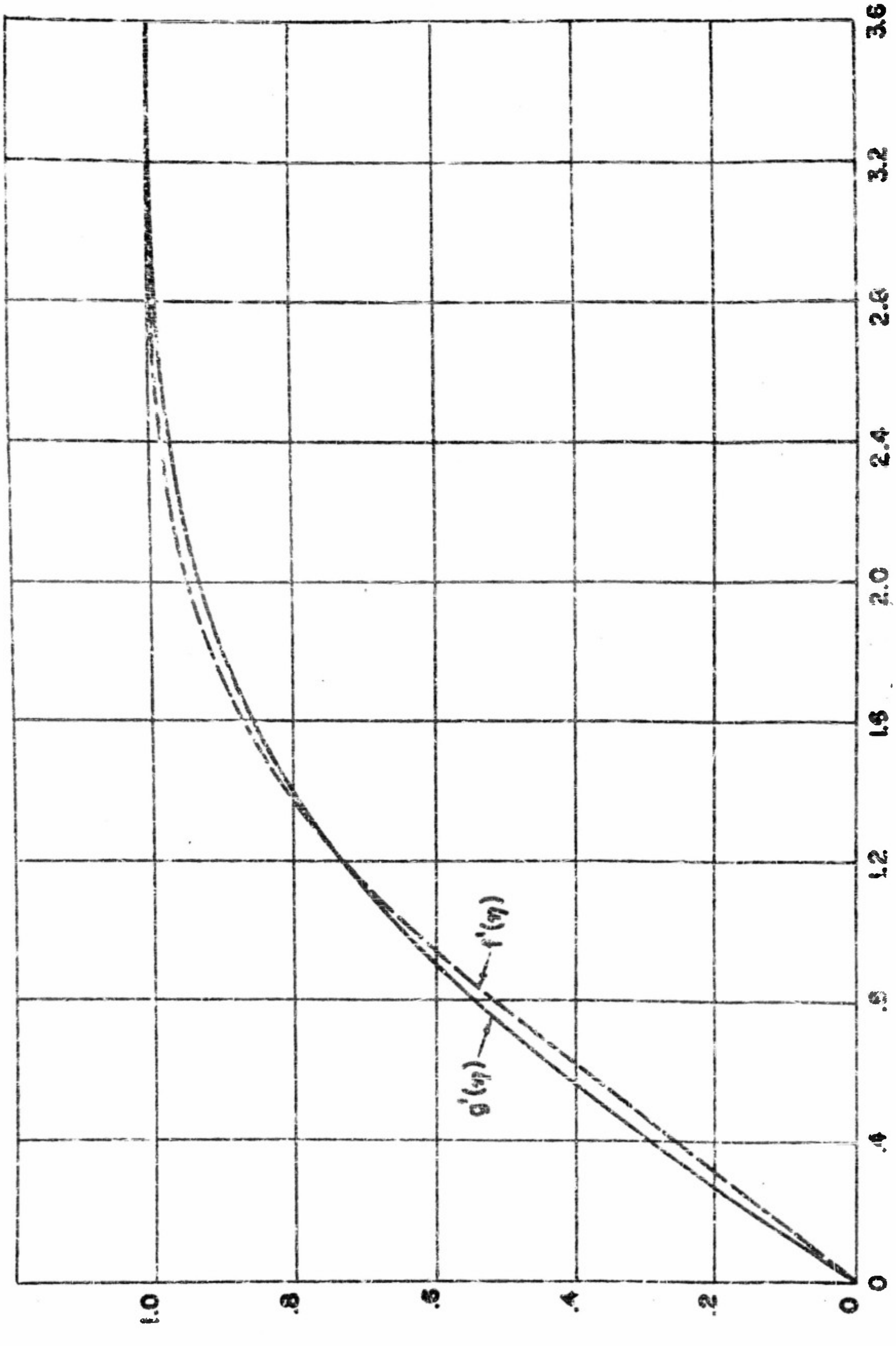

FIG. 6.1 Comparison of Blasius versus linearized analysis (C=.43)